ON A CERTAIN INTEGRAL IN THE THEORY OF A HORIZONTAL GYROCOMPASS

(OB ODNOM INTEGRALE V TEORII GIROGORIZONTKOMPASA)

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This paper presents the results of rigorous investigations (without using the precessional theory) of the motion of a horizontal gyrocompass [gyrohorizoncompass] of the Gekeler-Anschutz type [1]. The authors present the dynamic equations of the system, supplemented by the kinematic equations, which are analogous to the well known equations of Poisson.

These equations, together with the initial conditions, yield the first integral which turns out to be the generalization of the corresponding integral obtained from the precessional theory as shown in [2,3].

This integral is being used to obtain the conditions of stability of the unperturbed motion of the system.

1. Let $Ox^\circ y^\circ z^\circ$ and $Oxyz$ be right-handed coordinate systems whose origins coincide with the suspension point of the gyrosphere. In the first system the x°-axis is along the velocity vector of the suspension point, and the z°-axis is along the earth's radius, directed upwards. In the second system the y-axis is along the vector of the angular momentum, and the z-axis is parallel to the axis of the casings of the gyroscopes.

	x°	y°	z°
x	θ_1	ϑ_1	ψ_1
y	θ_2	ϑ_2	ψ_2
z	θ_3	ϑ_3	ψ_3

The orientation of the $Ox^\circ y^\circ z^\circ$ system with respect to the $Oxyz$ system is expressed through the angles α, β, γ.

The direction cosines of an axis of one of the systems with respect to another system (using the notation shown in the table) are

$$\theta_1 = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma, \qquad \theta_2 = -\sin\alpha\cos\beta,$$
$$\theta_3 = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma$$
$$\vartheta_1 = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma, \qquad \vartheta_2 = \cos\alpha\cos\beta,$$
$$\vartheta_3 = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$$
$$\psi_1 = -\cos\beta\sin\gamma, \qquad \psi_2 = \sin\beta, \qquad \psi_3 = \cos\beta\cos\gamma \tag{1.1}$$

The equations of motion of a gyrocompass along the x-, y- and z-axes are

$$\begin{aligned} A\frac{dp}{dt} + (C-B)qr - Hr &= -\left(F - m\frac{v^2}{R}\right)l\psi_2 - mvl\Omega\vartheta_2 - ml\frac{dv}{dt}\theta_2 \\ B\frac{dq}{dt} + (A-C)rp + \frac{dH}{dt} &= \left(F - m\frac{v^2}{R}\right)l\psi_1 + mvl\Omega\vartheta_1 + ml\frac{dv}{dt}\theta_1 \\ C\frac{dr}{dt} + (B-A)pq + Hp &= 0 \qquad (H = 2B'\cos\varepsilon) \\ -2I\frac{d^2\varepsilon}{dt^2} - 2B'\sin\varepsilon q &= N(\varepsilon) \end{aligned} \tag{1.2}$$

Here A, B, C are the principal moments of inertia about the axes x, y, z, respectively; I is the moment of inertia of the rotor and its casing about the axis of the casing; B' is the angular momentum of the gyroscope; p, q, r are the x, y, z components of the angular velocity vector of the xyz system; Ω is the angular velocity of the $x^\circ y^\circ z^\circ$ system about the z°-axis.

We have then according to [4]

$$\Omega = U\sin\varphi + \frac{v_E}{R}\tan\varphi + \frac{d\alpha^*}{dt} \qquad \left(\tan\alpha^* = \frac{v_N}{RU\cos\varphi + v_E}\right) \tag{1.3}$$

The remaining symbols in (1.2) and (1.3) have the same meaning as in [1,4]. The formulas for p, q, r have the following form [1]

$$\begin{aligned} p &= \frac{v}{R}\vartheta_1 + (\Omega + \dot\alpha)\psi_1 + \dot\beta\cos\gamma \\ q &= \frac{v}{R}\vartheta_2 + (\Omega + \dot\alpha)\psi_2 + \dot\gamma \\ r &= \frac{v}{R}\vartheta_3 + (\Omega + \dot\alpha)\psi_3 + \dot\beta\sin\gamma \end{aligned} \tag{1.4}$$

2. Let Ω_0 and Ω_1 be the angular velocity vectors of the systems $x^\circ y^\circ z^\circ$ and xyz respectively, and let the unit vectors along the axes x°, y°, z°, and x, y, z be $\mathbf{i}^\circ$, $\mathbf{j}^\circ$, $\mathbf{k}^\circ$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively. Then [1]

$$\boldsymbol{\Omega}_0 = \mathbf{j}^\circ\frac{v}{R} + \mathbf{k}^\circ\Omega, \qquad \boldsymbol{\Omega}_1 = \mathbf{i}p + \mathbf{j}q + \mathbf{k}r \tag{2.1}$$

We have, obviously, the following equations

$$\frac{d\mathbf{i}^\circ}{dt} = \mathbf{\Omega}_0 \times \mathbf{i}^\circ = \left(\frac{d\mathbf{i}^\circ}{dt}\right)_{xyz} + \mathbf{\Omega}_1 \times \mathbf{i}^\circ \qquad (\mathbf{i}^\circ\mathbf{j}^\circ\mathbf{k}^\circ) \tag{2.2}$$

Here the indices xyz refer to the local derivative, while the symbols $(\mathbf{i}^\circ\mathbf{j}^\circ\mathbf{k}^\circ)$ mean that the remaining equations are obtained by cyclic permutations. Further

$$\mathbf{i}^\circ = \mathbf{i}\theta_1 + \mathbf{j}\theta_2 + \mathbf{k}\theta_3, \qquad \mathbf{j}^\circ = \mathbf{i}\vartheta_1 + \mathbf{j}\vartheta_2 + \mathbf{k}\vartheta_3, \qquad \mathbf{k}^\circ = \mathbf{i}\psi_1 + \mathbf{j}\psi_2 + \mathbf{k}\psi_3$$

Taking into account that

$$\mathbf{\Omega}_0 \times \mathbf{i}^\circ = \mathbf{j}^\circ\Omega - \mathbf{k}^\circ \frac{v}{R}, \qquad \mathbf{\Omega}_0 \times \mathbf{j}^\circ = -\mathbf{i}^\circ\Omega, \qquad \mathbf{\Omega}_0 \times \mathbf{k}^\circ = \mathbf{i}^\circ \frac{v}{R}$$

we obtain the generalized Poisson equations in the form

$$\frac{d\theta_1}{dt} + q\theta_3 - r\theta_2 = \Omega\vartheta_1 - \frac{v}{R}\psi_1 \qquad (123,\ pqr) \tag{2.3}$$

$$\frac{d\vartheta_1}{dt} + q\vartheta_3 - r\vartheta_2 = -\Omega\theta_1 \qquad (123,\ pqr) \tag{2.4}$$

$$\frac{d\psi_1}{dt} + q\psi_3 - r\psi_2 = \frac{v}{R}\theta_1 \qquad (123,\ pqr) \tag{2.5}$$

Here the symbols $(123,\ pqr)$ mean that the remaining equations are obtained by cyclic permutation.

3. Let v and Ω be constants, and let $N(\varepsilon)$ satisfy the condition [1]

$$N(\varepsilon) = -\frac{4B'^2}{mlR}\sin\varepsilon\cos\varepsilon \tag{3.1}$$

Then, according to the last equation in the system (1.2), we have with accuracy up to the term $-2Id^2\varepsilon/dt^2$

$$2B'\cos\varepsilon = mlRq \tag{3.2}$$

Let us mention that if in the system which controls the moment $N(\varepsilon)$ through a transmitter the second derivative of the signal is obtainable, and the signal is proportional to the instantaneous value of the angle ε, then the equation (3.2) can be obtained without neglecting the term $-2Id^2\varepsilon/dt^2$. In this case $N(\varepsilon)$ must satisfy the condition

$$N(\varepsilon) = -\frac{4B'^2}{mlR}\sin\varepsilon\cos\varepsilon - 2I\frac{d^2\varepsilon}{dt^2} \tag{3.3}$$

If the moment is controlled by a transmitter according to the formula (3.3), then an investigation of stability of the control system itself is necessary.

Using formula (3.2) we obtain the equations (1.2) in the form

$$A\frac{dp}{dt}+(C-B_1)qr=-\left(F-m\frac{v^2}{R}\right)l\psi_2-mvl\Omega\vartheta_2$$

$$B_1\frac{dq}{dt}+(A-C)rp=\left(F-m\frac{v^2}{R}\right)l\psi_1+mvl\Omega\vartheta_1 \tag{3.4}$$

$$C\frac{dr}{dt}+(B_1-A)pq=0 \qquad (B_1=B+mlR)$$

4. To obtain the first integral we multiply the right and the left members of the three equations (3.4) by p, q, r respectively.

Adding together the multiplied equations we obtain

$$\frac{1}{2}A\frac{d}{dt}p^2+\frac{1}{2}B_1\frac{d}{dt}q^2+\frac{1}{2}C\frac{d}{dt}r^2=$$
$$=\left(F-m\frac{v^2}{R}\right)l(q\psi_1-p\psi_2)+mvl\Omega(q\vartheta_1-p\vartheta_2)$$

which, on taking into account the kinetic equations (2.4) and (2.5) becomes

$$\frac{1}{2}A\frac{d}{dt}p^2+\frac{1}{2}B_1\frac{d}{dt}q^2+\frac{1}{2}C\frac{d}{dt}r^2-\left(F-m\frac{v^2}{R}\right)l\frac{d\psi_3}{dt}-mvl\Omega\frac{d\vartheta_3}{dt}=$$
$$=-\frac{v}{R}\left(F-m\frac{v^2}{R}-mR\Omega^2\right)l\theta_3 \tag{4.1}$$

We shall multiply now the equations (3.4) and (2.5) by ψ_1, ψ_2, ψ_3 and Ap, B_1q, Cr respectively.

Adding together the left and the right members and taking into account that $\psi_2\vartheta_1-\psi_1\vartheta_2=\theta_3$, we obtain

$$A\frac{d}{dt}(p\psi_1)+B_1\frac{d}{dt}(q\psi_2)+C\frac{d}{dt}(r\psi_3)=$$
$$=mvl\Omega\theta_3+\frac{v}{R}(Ap\theta_1+B_1q\theta_2+Cr\theta_3) \tag{4.2}$$

Next we shall multiply the equations (3.4) and (2.4) by ϑ_1, ϑ_2, ϑ_3, and Ap, B_1q, Cr respectively, add them together and obtain also

$$A\frac{d}{dt}(p\vartheta_1)+B_1\frac{d}{dt}(q\vartheta_2)+C\frac{d}{dt}(r\vartheta_3)=$$
$$=-\left(F-m\frac{v^2}{R}\right)l\theta_3-\Omega(Ap\theta_1+B_1q\theta_2+Cr\theta_3) \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$\left[A \frac{d}{dt}(p\psi_1) + B_1 \frac{d}{dt}(q\psi_2) + C \frac{d}{dt}(r\psi_3)\right]\Omega +$$
$$+\left[A \frac{d}{dt}(p\vartheta_1) + B_1 \frac{d}{dt}(q\vartheta_2) + C \frac{d}{dt}(r\vartheta_3)\right]\frac{v}{R} =$$
$$= -\frac{v}{R}\left(F - m\frac{v^2}{R} - mR\Omega^2\right) l\theta_3 \qquad (4.4)$$

Combining the above expression with (4.1) we obtain the integrable equation

$$\frac{1}{2} A \frac{d}{dt} p^2 + \frac{1}{2} B_1 \frac{d}{dt} q^2 + \frac{1}{2} C \frac{d}{dt} r^2 - \left(F - m\frac{v^2}{R}\right) l \frac{d\psi_3}{dt} - mvl\Omega \frac{d\vartheta_3}{dt} =$$
$$= \left[A \frac{d}{dt}(p\psi_1) + B_1 \frac{d}{dt}(q\psi_2) + C \frac{d}{dt}(r\psi_3)\right]\Omega +$$
$$+\left[A \frac{d}{dt}(p\vartheta_1) + B_1 \frac{d}{dt}(q\vartheta_2) + C \frac{d}{dt}(r\vartheta_3)\right]\frac{v}{R} \qquad (4.5)$$

which yields the sought first integral

$$V \equiv \frac{1}{2}(Ap^2 + B_1 q^2 + Cr^2) - \left(F - m\frac{v^2}{R}\right) l\psi_3 - mvl\Omega\vartheta_3 -$$
$$-(Ap\psi_1 + B_1 q\psi_2 + Cr\psi_3)\Omega - (Ap\vartheta_1 + B_1 q\vartheta_2 + Cr\vartheta_3)\frac{v}{R} = C_1 \qquad (4.6)$$

Let us mention that if we neglect the inertial terms containing the moments of inertia A, B, C as multipliers, then with accuracy to constant terms the formula (4.6) reduces to the integral obtained by Merkin in [2].

5. The integral (4.6) can be used for the analysis of stability of motion of a horizontal gyrocompass. For this purpose let us turn back to the equations (3.4) and to the formulas (1.1) and (1.4). From them it follows that the position of equilibrium of the system occurs at the following values of the coordinates α, β, γ

$$\alpha = 0, \qquad \beta = \beta^*, \qquad \gamma = 0 \qquad (5.1)$$

and further β^* satisfies the equation

$$(C - B_1)\left\{\frac{1}{2}\left[\Omega^2 - \left(\frac{v}{R}\right)^2\right]\sin 2\beta^* + \frac{v}{R}\Omega\cos 2\beta^*\right\} =$$
$$= -\left(F - m\frac{v^2}{R}\right) l \sin\beta^* - mvl\Omega\cos\beta^* \qquad (5.2)$$

The motion described by the equation (5.1) is assumed to be unperturbed. In the perturbed motion we shall set

$$\alpha = x_1, \qquad \beta = \beta^* + x_2, \qquad \gamma = x_3 \tag{5.3}$$

We shall denote by V_0 the value of the function V when $x_1 = x_2 = x_3 = 0$ and $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$, and construct the difference $V - V_0$, obtaining

$$\begin{aligned} V - V_0 = {} & \frac{1}{2}(B_1 \sin^2\beta^* + C\cos^2\beta^*)\dot{x}_1^2 + \frac{1}{2}A\dot{x}_2^2 + \frac{1}{2}B_1\dot{x}_3^2 + \\ & + B_1 \sin\beta^* \dot{x}_1\dot{x}_3 + \frac{1}{2}\frac{v}{R}\Big[-A\frac{v}{R} + B_1\cos\beta^*\Big(\frac{v}{R}\cos\beta^* + \Omega\sin\beta^*\Big) + \\ & + C\sin\beta^*\Big(\frac{v}{R}\sin\beta^* - \Omega\cos\beta^*\Big) - mlR\Omega\sin\beta^*\Big]x_1^2 + \\ & + \frac{1}{2}\Big\{(C - B_1)\Big[\Big(\Omega\cos\beta^* - \frac{v}{R}\sin\beta^*\Big)^2 - \Big(\Omega\sin\beta^* + \frac{v}{R}\cos\beta^*\Big)^2\Big] + \\ & + \Big(F - m\frac{v^2}{R}\Big)l\cos\beta^* - mvl\Omega\sin\beta^*\Big\}x_2^2 + \\ & + \frac{1}{2}\Big[(C - A)\Big(\Omega\cos\beta^* - \frac{v\sin\beta^*}{R}\Big)^2 + \Big(F - \frac{mv^2}{R}\Big)l\cos\beta^* - mvl\Omega\sin\beta^*\Big]x_3^2 + \\ & + \frac{v}{R}\Big[(C - A)\Big(\frac{v}{R}\sin\beta^* - \Omega\cos\beta^*\Big) - mlR\Omega\Big]x_1x_3 + \cdots \end{aligned} \tag{5.4}$$

where the dots denote higher order terms of x_s and $\dot{x}_s$, which were neglected.

The expression (5.4) can be simplified if we keep only second order terms of x_s, β^* and of the corresponding derivatives. Then

$$\begin{aligned} V - V_0 = {} & \frac{1}{2}C\dot{x}_1^2 + \frac{1}{2}A\dot{x}_2^2 + \frac{1}{2}B_1\dot{x}_3^2 + \frac{1}{2}(B_1 - A)\Big(\frac{v}{R}\Big)^2 x_1^2 + \\ & + \frac{1}{2}\Big\{(C - B_1)\Big[\Omega^2 - \Big(\frac{v}{R}\Big)^2\Big] + \Big(F - m\frac{v^2}{R}\Big)l\Big\}x_2^2 + \\ & + \frac{1}{2}\Big[(C - A)\Omega^2 + \Big(F - m\frac{v^2}{R}\Big)l\Big]x_3^2 - \Big[mvl\Omega + (C - A)\frac{v}{R}\Omega\Big]x_1x_3 + \cdots \end{aligned} \tag{5.5}$$

Noticing that the first three terms in (5.5) represent a positive-definite quadratic form we shall consider the form

$$\begin{aligned} W \equiv {} & \frac{1}{2}(B_1 - A)\Big(\frac{v}{R}\Big)^2 x_1^2 + \frac{1}{2}\Big\{\Big(F - m\frac{v^2}{R}\Big)l + (C - B_1)\Big[\Omega^2 - \Big(\frac{v}{R}\Big)^2\Big]\Big\}x_2^2 + \\ & + \frac{1}{2}\Big[\Big(F - m\frac{v^2}{R}\Big)l + (C - A)\Omega^2\Big]x_3^2 - \Big[mvl\Omega + (C - A)\frac{v}{R}\Omega\Big]x_1x_3 \end{aligned} \tag{5.6}$$

Applying to (5.6) the criterium of Sylvester we find out that W would be positive-definite at sufficiently small values of x_s if the following inequalities are satisfied

$$B - A + mlR > 0, \qquad Fl - mlR\Omega^2 + (C - B)\Big[\Omega^2 - \Big(\frac{v}{R}\Big)^2\Big] > 0 \tag{5.7}$$

$$\left(F - m\frac{v^2}{R}\right)\left(1 + \frac{B-A}{mlR}\right) - mR\Omega^2\left[1 + \frac{C-A}{mlR}\left(1 + \frac{C-B}{mlR}\right)\right] > 0 \quad (5.7)$$

Under the conditions (5.7) the form $V - V_0$ would also be positive-definite.

Since on the strength of the equations of the perturbed motion the total derivative of the above form equals identically zero, the unperturbed motion (5.1), according to Liapunov, is stable.

The sufficient conditions of stability (5.7) permit degeneration which makes them applicable also for the precessional theory. If in (5.7) we neglect the terms containing A, B, C as multipliers we obtain the inequality

$$F - m\frac{v^2}{R} - mR\Omega^2 > 0 \quad (5.8)$$

which has been established previously in [2].

The above condition follows also from (5.7) in the case of a full kinetic symmetry, that is when $A = B = C$.

If the centrifugal acceleration v^2/R is added to the gravitational acceleration, then from (5.8) we obtain the inequality

$$\Omega < \nu \qquad (\nu = \sqrt{g/R}) \quad (5.9)$$

which has been obtained previously in [2-4].

It has been shown in [2] (in the precessional case and with small dissipative forces), that the inequality (5.9) is not only the sufficient but also the necessary condition of stability. The reasoning which has been used in [2] is also applicable to our case.

6. Let us consider the transcendental equation (5.2) from which we can easily obtain an approximate expression for β^* in the case when the base is stationary. Assuming that β^* is small and assuming also that $F - mv^2/R \approx mg$, we have

$$\beta^* = -\frac{(C-B)U^2\sin\varphi\cos\varphi}{mgl - (C-B_1)U^2\cos 2\varphi} \quad (6.1)$$

Taking into account that $mgl \gg (C - B_1)U^2 \cos 2\varphi$, we obtain from (6.1)

$$\beta^* = -\frac{C-B}{mlR}\beta_s \qquad \left(\beta_s = \frac{1}{2}\frac{U^2\sin 2\varphi}{\nu^2}\right) \quad (6.2)$$

The quantity β^* is measured in fractions of an arc second and is of no practical importance.

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